Announcements

1) Entering a vector into webwork: (# 7)

$\langle 5, 2 \rangle$ is an example

2) Careful with little errors!

3) New HW - appearing later today, due next Wednesday

Example 1: Describe all solutions to

$$5x_1 - 2x_2 + x_3 - 6x_4 = 1$$
$$-x_1 + 6x_2 + 10x_3 + 2x_4 = -1$$

Matrix

$$\begin{bmatrix} 5 & -2 & 1 & -6 & 1 \\ -1 & 6 & 10 & 2 & -1 \end{bmatrix}$$

rref

$$\begin{bmatrix} 1 & 0 & 13/14 & -8/7 & 1/7 \\ 0 & 1 & 51/28 & 1/7 & -1/7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 13/14 & -8/7 & 1/7 \\ 0 & 1 & 51/28 & 1/7 & -1/7 \end{bmatrix}$$

what does this say?

$$x_1 + 13/14\, x_3 - 8/7\, x_4 = 1/7$$

$$x_2 + 51/28\, x_3 + 1/7\, x_4 = -1/7$$

So

$$x_1 = 1/7 - 13/14\, x_3 + 8/7\, x_4$$

$$x_2 = -51/28\, x_3 - 1/7\, x_4 - 1/7$$

The most general solution is then

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1/7 - 13/14\, x_3 + 8/7\, x_4 \\ -1/7 - 5/28\, x_3 - 1/7\, x_4 \\ x_3 \\ x_4 \end{bmatrix}$$

$$= \begin{bmatrix} 1/7 \\ -1/7 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -13/14 \\ -5/28 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 8/7 \\ -1/7 \\ 0 \\ 1 \end{bmatrix}$$

How does the row-reduced matrix correspond to the three cases for solutions?

1) No solution: you will see a row of all zeros except for the last entry

e.g.
$$\underbrace{\begin{bmatrix} 1 & 0 & 6 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{no solution}}$$

2) Unique solution:

every row either is all zeros or has exactly one nonzero entry (which is a one) in all columns to the left of the last one.

e.g.

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{unique solution}}$$

3) Infinitely many solutions

any other situation! In particular, you will have no rows that are all zeros except for the last number and at least one row will have two nonzero entries to the left of the last column.

e.g. example 1 today or

$$\underbrace{\begin{bmatrix} 1 & 0 & 5 & 3 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{infinitely many solutions}}$$

How to get solutions faster

Homogeneous equations

A linear equation $Ax = b$ is called homogeneous if b is the zero vector (all coordinates zero).

Such an equation always has at least one solution, which is the trivial solution where x is the zero vector.

Example 2:

$$\begin{bmatrix} -1 & 3 & 0 \\ 6 & 8 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

is a homogeneous equation.

Note the solution $x_1 = x_2 = x_3 = 0$.

But

$$\begin{bmatrix} 3 & 18 \\ -\pi & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

is <u>not</u> homogeneous.

Briefly, if A is an $m \times n$ matrix, x is an n-vector, and $\vec{0}$ denotes the m-vector with zeroes in all entries, then a homogeneous equation is of the form

$$\boxed{Ax = \vec{0}}$$

Given vectors $v_1, v_2, \ldots, v_k$ in $\mathbb{R}^n$, we define a linear combination of $v_1, v_2, \ldots, v_k$ to be any vector of the form

$$\boxed{a_1 v_1 + a_2 v_2 + \cdots + a_k v_k}$$

where $a_1, a_2, \ldots, a_k$ are numbers.

Example 3:

Let $v_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

and $v_2 = \begin{bmatrix} -1 \\ 8 \end{bmatrix}$.

The vector

$v = \begin{bmatrix} 0 \\ 28 \end{bmatrix}$ is a linear

combination of v_1 and v_2 since

$$v_1 + 3v_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix} + \begin{bmatrix} -3 \\ 24 \end{bmatrix} = \begin{bmatrix} 0 \\ 28 \end{bmatrix}$$

Span: Given vectors $v_1, v_2, \ldots, v_k$ in $\mathbb{R}^n$, the **span** of $v_1, v_2, \ldots, v_k$ is equal to all vectors v that are linear combinations of $v_1, v_2, \ldots, v_k$.

Example 4: Describe the span of

$$v_1 = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} \text{ and } v_2 = \begin{bmatrix} 6 \\ 0 \\ -2 \end{bmatrix}$$

in $\mathbb{R}^3$.

The span is all vectors of the form

$a_1 v_1 + a_2 v_2$ for numbers a_1 and a_2.

We don't get all of $\mathbb{R}^3$ since we don't have enough vectors.

The span is a plane since we have two vectors and neither one is a multiple of the other.

Moreover, the plane goes through the origin since we can choose $a_1 = a_2 = 0$ and get $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ in the span.

Observations:

1) The zero vector is always in the span.

2) The vectors $v_1, v_2, \ldots, v_k$ are always in their span.

So the span of any nonzero vectors is always infinite!